

# Leavitt path algebras with finitely presented irreducible representations

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## Abstract

Let  $E$  be an arbitrary graph,  $K$  be any field and let  $L = L_K(E)$  be the corresponding Leavitt path algebra. Necessary and sufficient conditions (both graphical and algebraic) are given under which all the irreducible representations of  $L$  are finitely presented. In this case, the graph  $E$  turns out to be row-finite and the cycles in  $E$  form an artinian partial ordered set under a defined relation  $\geq$ . When the graph  $E$  is finite, the above graphical conditions were shown in [7] to be equivalent to  $L_K(E)$  having finite Gelfand-Kirillov dimension. Examples show that this equivalence no longer holds for infinite graphs and a complete description is obtained of Leavitt path algebras over arbitrary graphs having finite Gelfand-Kirillov dimensions.

## 1 Introduction and Preliminaries

The notion of Leavitt path algebras was introduced and initially studied in [1], [11] as algebraic analogues of graph  $C^*$ -algebras and as the natural generalization of the Leavitt algebras of type  $(1, n)$  built in [23]. The module theory over Leavitt path algebras was initiated in [9] and in other recent papers ([12], [13], [24]). In [19], Goncalves and Royer indicated a method of constructing various representations of a Leavitt path algebra  $L_K(E)$  over a graph  $E$  by using the concept of algebraic branching systems. Expanding this, Chen [17] studied special types of irreducible representations of  $L_K(E)$  induced by the sinks as well as the equivalence class  $[p]$  of infinite paths tail-equivalent (see definition below) to a fixed infinite path  $p$  in  $E$  and he further noted that these can also be considered as algebraic branching systems. Additional ways of constructing irreducible representations of  $L_K(E)$  were pointed out in [12] while in [24] a new

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\*2010 *Mathematics Subject Classification*: 16D70; *Key words and phrases*: Leavitt path algebras, arbitrary graphs, simple modules, finitely presented modules, Gelfand-Kirillov dimension.

class  $\mathbf{S}_v$  of irreducible representations was constructed using vertices  $v$  which emit infinitely many edges.

When  $E$  is a finite graph, it was shown in [12] that every simple left module over  $L_K(E)$  is finitely presented if and only if every vertex in  $E$  is the base of at most one cycle. In this paper, we wish to extend this theorem to the case when  $E$  is an arbitrary graph. Unlike the case of finite graphs, the existence of infinite paths and vertices emitting infinitely many edges are to be dealt with appropriately. We first show that if a vertex  $v$  in graph  $E$  emits infinitely many edges, then the corresponding simple module  $\mathbf{S}_v$  defined in [24] is not finitely presented. Thus the graph  $E$  must be row-finite if every simple left module over  $L_K(E)$  is to be finitely presented. Generalization of a Lemma from [12] to the case of row-finite graphs shows that simple modules induced by infinite irrational paths not containing a line point (see definition below) are also not finitely presented. Eliminating these possibilities, we are able to obtain a complete characterization (both graphical and algebraic) of Leavitt path algebras  $L_K(E)$  over which every simple left/right module is finitely presented, thus leading to the following main theorem. Here we denote a pre-order  $\geq$  among cycles by writing  $c \geq c'$  for two cycles  $c$  and  $c'$  if there is a path connecting a vertex on  $c$  to a vertex on  $c'$ .

**Theorem 1.1** *Let  $E$  be an arbitrary graph,  $K$  be any field and let  $L = L_K(E)$ . Then the following statements are equivalent:*

- (1) *Every simple left/right  $L$ -module is finitely presented;*
- (2)  *$L$  is the union of a continuous well-ordered ascending chain of graded ideals*

$$0 \leq I_1 < \cdots < I_\alpha < I_{\alpha+1} < \cdots \quad (\alpha < \tau) \quad (***)$$

where  $\tau$  is a suitable ordinal,  $I_1 = \text{Soc}(L)$  and, for each  $\alpha \geq 1$  with  $I_\alpha \neq L$ ,  $I_{\alpha+1}/I_\alpha \cong M_{\Lambda_\alpha}(K[x, x^{-1}])$ , where  $\Lambda_\alpha$  is an arbitrary index set (depending on  $\alpha$ ).

- (3)  *$E$  is row-finite, and either (a)  $E^0$  is the saturated closure of the set of all line points in  $E$  (and is, in particular, acyclic) or (b)(i)  $E$  contains cycles and the set  $C$  of all the cycles in  $E$  becomes an artinian partially ordered set under the relation  $\geq$ , (ii) every infinite path in  $E$  either contains a line point or is tail equivalent to a rational path and (iii) For every proper hereditary saturated subset of vertices  $H$  containing all the line points in  $E$ ,  $E \setminus H$  contains cycles without exits but does not contain any line points.*

Observing that, for a finite graph  $E$ , Condition (3)(b)(i) of the above theorem is equivalent to the condition that distinct cycles in  $E$  have no common vertex and that Conditions (3)(b)(ii) and (iii) are automatically satisfied, we obtain the main theorem of [12]:

**Corollary 1.2** [12] *If  $E$  is a finite graph, then every simple left module over  $L_K(E)$  is finitely presented if and only if distinct cycles in  $E$  are disjoint, that is, have no common vertex.*

Interestingly, the graphical condition for a finite graph  $E$  in the preceding corollary (that distinct cycles in  $E$  have no common vertex) has been shown in [7] to be equivalent to the condition that the corresponding Leavitt path algebra  $L_K(E)$  has finite Gelfand-Kirillov dimension (for short, GK-dimension). A natural question is whether this equivalence extends to arbitrary graphs. After constructing examples showing that this equivalence no longer holds for infinite graphs, use of a direct limit construction done in [5] leads to an easy extension of the result of [7] to arbitrary graphs (Theorem 5.2). These algebras seem to be "made up" of von Neumann regular rings and the Laurent polynomial ring  $K[x, x^{-1}]$ .

A (directed) graph  $E = (E^0, E^1, r, s)$  consists of two sets  $E^0$  and  $E^1$  together with maps  $r, s : E^1 \rightarrow E^0$ . The elements of  $E^0$  are called *vertices* and the elements of  $E^1$  *edges*. We generally follow the notation, terminology and results from [2], [1] and [11]. We outline some of the concepts and results that will be used in this paper.

A vertex  $v$  is called a *sink* if it emits no edges, that is,  $s^{-1}(v) = \emptyset$ , the empty set. The vertex  $v$  is called a *regular vertex* if  $s^{-1}(v)$  is finite and non-empty and  $v$  is called an *infinite emitter* if  $s^{-1}(v)$  is infinite. For each  $e \in E^1$ , we call  $e^*$  a ghost edge. We let  $r(e^*)$  denote  $s(e)$ , and we let  $s(e^*)$  denote  $r(e)$ . A *finite path*  $\mu$  of length  $n > 0$  is a finite sequence of edges  $\mu = e_1 e_2 \cdots e_n$  with  $r(e_i) = s(e_{i+1})$  for all  $i = 1, \dots, n-1$ . In this case  $\mu^* = e_n^* \cdots e_2^* e_1^*$  is the corresponding ghost path. Any vertex  $v$  is considered a path of length 0. The set of all vertices on a path  $\mu$  is denoted by  $\mu^0$ .

Given an arbitrary graph  $E$  and a field  $K$ , the *Leavitt path algebra*  $L_K(E)$  is defined to be the  $K$ -algebra generated by a set  $\{v : v \in E^0\}$  of pairwise orthogonal idempotents together with a set of variables  $\{e, e^* : e \in E^1\}$  which satisfy the following conditions:

- (1)  $s(e)e = e = er(e)$  for all  $e \in E^1$ .
- (2)  $r(e)e^* = e^* = e^*s(e)$  for all  $e \in E^1$ .
- (3) (The "CK-1 relations") For all  $e, f \in E^1$ ,  $e^*e = r(e)$  and  $e^*f = 0$  if  $e \neq f$ .
- (4) (The "CK-2 relations") For every regular vertex  $v \in E^0$ ,

$$v = \sum_{e \in E^1, s(e)=v} ee^*.$$

A subgraph  $F$  of a graph  $E$  is called a *complete subgraph* if, for any vertex  $v \in F$ ,  $s_F^{-1}(v) = s_E^{-1}(v)$ . In this case the subalgebra generated by  $F$  is isomorphic to  $L_K(F)$ .

A path  $\mu = e_1 \dots e_n$  in  $E$  is *closed* if  $r(e_n) = s(e_1)$ , in which case  $\mu$  is said to be *based at the vertex*  $s(e_1)$ . A closed path  $\mu$  as above is called *simple* provided it does not pass through its base more than once, i.e.,  $s(e_i) \neq s(e_1)$  for all  $i = 2, \dots, n$ . The closed path  $\mu$  is called a *cycle* if it does not pass through any of its vertices twice, that is, if  $s(e_i) \neq s(e_j)$  for every  $i \neq j$ . An *exit* for a path  $\mu = e_1 \dots e_n$  is an edge  $f$  such that  $s(f) = s(e_i)$  for some  $i$  and  $f \neq e_i$ . The graph  $E$  is said to satisfy *Condition (L)* if every closed path has an exit.

$E$  is said to satisfy *Condition (K)* if each vertex in  $E$  is the base of either no closed path or at least two distinct closed paths. Condition (K) always implies Condition (L).

A subset  $H$  of  $E^0$  is called *hereditary* if, whenever  $v \in H$  and there is a path from  $v$  to  $w \in E^0$ , then  $w \in H$ . A hereditary set is *saturated* if, for any regular vertex  $v$ ,  $r(s^{-1}(v)) \subseteq H$  implies  $v \in H$ . If  $E$  is row-finite and  $I$  is the ideal generated by a hereditary saturated set  $H$  of vertices, then  $L/I \cong L_K(E \setminus H)$  where  $E \setminus H$  is the "quotient graph" defined by setting  $(E \setminus H)^0 = E^0 \setminus H$  and  $(E \setminus H)^1 = \{e \in E^1 : r(e) \notin H\}$ , and the maps  $r, s$  are the same (see, [11]). Moreover, every element of  $I$  is a  $K$ -linear combination of monomials of the form  $pq^*$  where  $r(p) = r(q) \in H$ .

We shall also be using the following concepts and results from [27].

Let  $E$  be an arbitrary graph and let  $H$  be a hereditary saturated subset of vertices in  $E$ . An infinite emitter  $v$  is called a *breaking vertex* for  $H$  if  $0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty$ . The set of all breaking vertices for  $H$  is denoted by  $B_H$ . If  $v \in B_H$ , the  $v^H$  denotes the element  $v - \sum_{e \in s^{-1}(v), r(e) \notin H} ee^*$ . If  $I$  a

graded ideal of  $L_K(E)$  with  $I \cap E^0 = H$  and  $S = \{v \in B_H : v^H \in I\}$ , then it was shown in [27] that  $I$  is the ideal generated by  $H \cup \{v^H : v \in S\}$  and is denoted by  $I(H, S)$ . It was also shown in [27] that  $L_K(E)/I(H, S) \cong L_K(E \setminus (H, S))$  where  $E \setminus (H, S)$  is the quotient graph given by

$$(E \setminus (H, S))^0 := E^0 \setminus H \cup \{u' : u \in B_H \setminus S\};$$

$$(E \setminus (H, S))^1 := \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \setminus S\}.$$

Here  $r$  and  $s$  are extended to  $(E \setminus (H, S))^0$  by setting  $s(e') = s(e)$  and  $r(e') = r(e)'$ . We shall use the fact that  $u'$  is a sink for each  $u \in B_H \setminus S$ .

Given an infinite path  $p = e_1 e_2 \cdots e_n \cdots$  and an integer  $n \geq 1$ , Chen ([17]) defines  $\tau_{\leq n}(p) = e_1 \cdots e_n$  and  $\tau_{> n}(p) = e_{n+1} e_{n+2} \cdots$ . Two infinite paths  $p, q$  are said to be *tail-equivalent* if there exist positive integers  $m, n$  such that  $\tau_{> m}(p) = \tau_{> n}(q)$ . This is an equivalence relation and the equivalence class of all paths tail equivalent to an infinite path  $p$  is denoted by  $[p]$ . An infinite path  $p$  is called a *rational path* if  $p = ggg \cdots$  where  $g$  is some (finite) closed path in  $E$ . Given an infinite path  $p$ , Chen defines  $V_{[p]} = \bigoplus_{q \in [p]} Kq$ , a  $K$ -vector space having

$\{q : q \in [p]\}$  as a basis.  $V_{[p]}$  is made a left  $L$ -module by defining the module operation  $\cdot$ , for all  $q \in [p]$  and all  $v \in E^0$ ,  $e \in E^1$ , as follows:

- 1)  $v \cdot q = q$  or 0 according as  $v = s(q)$  or not;
- 2)  $e \cdot q = eq$  or 0 according as  $r(e) = s(q)$  or not;
- 3)  $e^* \cdot q = \tau_{> 1}(q)$  or 0 according as  $q = eq'$  or not.

In [17], Chen shows that under the above action of  $L$ ,  $V_{[p]}$  becomes a simple left  $L$ -module which we shall call a Chen simple module.

Following Chen, it was shown in [24] that if a vertex  $v$  is an infinite emitter, then the  $K$ -vector space  $\mathbf{S}_v$  having as a basis the set  $B = \{p : p \text{ a path in } E \text{ with } r(p) = v\}$  can be made a simple  $L_K(E)$ -module where the multiplication operation  $\cdot$  on elements of  $B$  by elements of  $L_K(E)$  is induced by conditions 1), 2), 3) above plus the additional condition that  $e^* \cdot v = 0$  for all edges  $e \in E^1$ . In particular,  $\beta^* \cdot v = 0$  in  $\mathbf{S}_v$  for all paths  $\beta$  in  $E$ .

For any vertex  $v$  in  $E$ , the *tree* of  $v$  is denoted by  $T_E(v)$  and is defined as  $T_E(v) = \{w \in E^0 : \text{there is a path from } v \text{ to } w\}$ . We say there is a *bifurcation* at a vertex  $v$ , if  $v$  emits more than one edge. In a graph  $E$ , a vertex  $v$  is called a *line point* if there is no bifurcation or a cycle based at any vertex in  $T_E(v)$ . Thus, if  $v$  is a line point, there will be a single finite or infinite line segment  $\mu$  starting at  $v$  ( $\mu$  could just be  $v$ ) and any other path  $\alpha$  with  $s(\alpha) = v$  will just be an initial sub-segment of  $\mu$ . It was shown in [14] that  $v$  is a line point in  $E$  if and only if  $vL_K(E)$  (and likewise  $L_K(E)v$ ) is a simple left (right) ideal. Moreover, the ideal generated by all the line points in  $E$  is the socle of  $L_K(E)$ . If  $v$  is a line point, then it is clear that any  $w \in T_E(v)$  is also a line point.

Even though the Leavitt path algebra  $L_K(E)$  may not have the multiplicative identity 1, we shall write  $L_K(E)(1 - v)$  to denote the set  $\{x - xv : x \in L_K(E)\}$ . If  $v$  is an idempotent (in particular, a vertex), we then get a direct decomposition  $L_K(E) = L_K(E)v \oplus L_K(E)(1 - v)$ .

Recall that a ring  $R$  is von Neumann regular if for every element  $a \in R$ , there is a  $b \in R$  such that  $a = aba$ .

## 2 When the graph $E$ contains no cycles

In this section, we describe all the acyclic graphs  $E$  such that every simple left/right module over  $L_K(E)$  is finitely presented.

We begin with a useful Remark.

**Remark 2.1** *Let  $E$  be an arbitrary graph and  $K$  be any field. Let  $L = L_K(E)$  and let  $L^1 = L \times K$ , be the unitization of  $L$  where the addition in  $L^1$  is termwise and the multiplication is given by  $(a, r)(b, s) = (ab + rb + sa, rs)$ . Identifying  $L$  with the set  $\{(a, 0) : a \in L\}$ , we notice that  $L$  is an ideal of  $L^1$  and that  $L^1/L \cong K$ . So  $L$  is von Neumann regular if and only if  $L^1$  is.*

*Also if  $M$  is any left  $L$ -module that is unital (i.e.  $LM = M$ ), then  $M$  is also a left  $L^1$ -module. Because, for any  $x \in M$ , there is a local unit  $u \in L$  such that  $ux = x$  and so, for any  $r_1 \in L^1$ , we can define  $r_1x = r_1ux = (r_1u)x \in M$ . From Proposition 2.4 of [15], every projective left  $L$ -module is also a projective  $L^1$ -module.*

**Theorem 2.2** *Let  $E$  be an arbitrary acyclic graph,  $K$  be any field and  $L = L_K(E)$ . Then the following are equivalent:*

- (i) *Every simple left/right  $L$ -module is finitely presented;*
- (ii)  *$L = \text{Soc}(L)$  is a direct sum of simple left/right ideals;*
- (iii)  *$E^0$  is the saturated closure of the set of all line points in  $E$ .*

**Proof.** Assume (i). We wish to show that every simple left  $L$ -module  $S$  is projective. Since  $S$  is finitely presented, we can write  $S = P/N$  where  $P$  is a projective  $L$ -module and both  $P$  and  $N$  are finitely generated. Now  $E$  has no cycles. So  $L$  (and hence, its unitization  $L^1$ ) is von Neuman regular by [5] and Remark 2.1. Also, as noted above,  $P$  is also a projective left module over  $L^1$ . On the other hand, it is known (see Theorem 1.11, [20]) that every finitely

generated submodule of the projective  $L^1$ -module  $P$  is a direct summand as a left  $L^1$ -module and hence as a left  $L$ -module. This means that  $N$  is a direct summand of  $P$  and hence  $S$  is projective. From the conclusion that every simple left  $L$ -module is projective, one can then easily show (applying Zorn's Lemma to the direct sums of simple left ideals in  $L$ ) that  $L$  is a direct sum of simple left ideals (see Proposition 2.27, [15]). A similar argument and conclusion holds for right  $L$ -modules. This proves (ii).

Now (ii)  $\Leftrightarrow$  (iii) follows from the fact that  $\text{Soc}(L)$  is the ideal generated by the set of all line points in  $E$  (see [14]) and that  $\text{Soc}(L) \cap E^0$  is the saturated closure of the hereditary set of all line points in  $E$ .

Assume (ii). Since  $L = \text{Soc}(L)$  is a semisimple left/right  $L$ -module, every submodule of  $L$  and, in particular, every maximal submodule of  $L$  is a direct summand of  $L$ . Since  $L$  is itself projective (see [15]), then every simple left/right  $L$ -module is projective and so is finitely presented. This proves (i). ■

**Remark:** Since a line point is either a sink or a regular vertex, it clear from the definition of the saturated closure, that Condition (iii) (that  $E^0$  is the saturated closure of the set of line points) implies that every vertex  $u \in E^0$  is a regular vertex. Hence the graph  $E$  must be row-finite. As we shall see later, a direct argument shows that the same conclusion holds even when  $E$  contains cycles. Also Condition (iii) of Theorem 2.2 is equivalent to the statement that given any vertex  $v$  in  $E$ , there is an integer  $n > 0$  such that every path from  $v$  of length  $> n$  ends at a line point (see Lemma 1.4, [6]).

### 3 When the graph $E$ contains cycles

We begin with an easy Lemma.

**Lemma 3.1** *Let  $E$  be an arbitrary graph and  $L = L_K(E)$ . For every maximal left ideal  $M$  of  $L$ , there is exactly one vertex  $u \notin M$  such that  $M = (M \cap Lu) \oplus \bigoplus_{v \in E^0, v \neq u} Lv$ . Thus every simple left  $L$ -module will be isomorphic to  $Lu/N$  for some vertex  $u$  in  $L$  and a maximal submodule  $N$  of  $Lu$ . Similar statements hold for a maximal right ideal of  $L$ .*

**Proof.** Since  $L = \bigoplus_{v \in E^0} Lv$  and  $M \neq L$ , there is a  $u \in E^0 \setminus M$ . Note that  $M \cap Lu \subset Mu$  and so  $M \cap Lu = Mu$ , by maximality. Then writing for any  $m \in M$ ,  $m = mu + (m - mu)$ , we get  $M = Mu \oplus M(1 - u) \subset Mu \oplus L(1 - u)$ . By maximality,  $M = Mu \oplus L(1 - u) = (M \cap Lu) \oplus \bigoplus_{v \in E^0 \setminus \{u\}} Lv$ . ■

For convenience sake, hereafter we shall consider only left  $L$ -modules. By symmetry, all our results also hold for right  $L$ -modules.

If the simple module  $S = Lu/N$  is, in addition finitely presented, then  $S \cong P/M'$  where  $P$  is a finitely generated projective  $L$ -module and  $M'$  is finitely generated. In that case, by Schanuel's Lemma [22],  $N$  will also be finitely generated. So, for an arbitrary graph  $E$ , checking whether all the simple left  $L_K(E)$ -modules are finitely presented or not is equivalent to checking whether,

for every vertex  $u$ , every maximal submodule of  $L_K(E)u$  is finitely generated or not.

It was shown in [24] that every infinite emitter  $v$  gives rise to a simple  $L$ -module  $(\mathbf{S}_v, \cdot)$  which has as a  $K$ -basis the set of all the paths in  $E$  that end at  $v$  and  $\mathbf{S}_v$  has the  $L$ -module operation  $\cdot$  as indicated in the Preliminaries section. The next proposition shows that, for any infinite emitter  $v$ , the simple module  $\mathbf{S}_v$  is not finitely presented.

**Proposition 3.2** *Let  $E$  be an arbitrary graph. If  $v \in E^0$  is an infinite emitter, then the corresponding simple left module  $(\mathbf{S}_v, \cdot)$  over  $L = L_K(E)$  is not finitely presented.*

**Proof.** Suppose, on the contrary,  $\mathbf{S}_v$  is finitely presented. Writing  $\mathbf{S}_v = L \cdot v$ , consider the exact sequence

$$0 \longrightarrow M \longrightarrow L \xrightarrow{\theta} \mathbf{S}_v \longrightarrow 0$$

where  $\theta(a) = a \cdot v$  for all  $a \in L$ . By Lemma 3.1,  $M = (\bigoplus_{u \in E^0, u \neq v} Lu) \oplus N$  where  $N = M \cap Lv$ . Restricting  $\theta$  to  $Lv$  we get an exact sequence

$$0 \longrightarrow N \longrightarrow Lv \xrightarrow{\theta} \mathbf{S}_v \longrightarrow 0,$$

where  $N$  is a finitely generated left ideal by our supposition. Let  $x_1, \dots, x_k$  be the generators of  $N$ . For each  $r = 1, \dots, k$ , we can write  $x_r = \sum_{i=1}^{m_r} k_i \alpha_i \beta_i^*$  where  $m_r$  is some positive integer and, for all  $i = 1, \dots, m_r$ ,  $v = s(\beta_i)$  and  $r(\alpha_i) = r(\beta_i)$ . Now every non-zero term in the summation for  $x_r$  must involve a ghost path  $\beta_i^*$ . Because, otherwise, re-indexing the terms, we can write  $x_r = \sum_{i=1}^s k_i \alpha_i + \sum_{j=s+1}^{m_r} k_j \alpha_j \beta_j^*$  where, for all  $i = 1, \dots, s$ , we assume that the real paths  $\alpha_i$  are all different, that  $k_i \neq 0$  and that  $r(\alpha_i) = v$  (as  $x_r v = x_r$ ). Then, since  $\beta_j^* \cdot v = 0$  for all  $j$ , we obtain  $0 = \theta(x_r) = x_r \cdot v = (\sum_{i=1}^r k_i \alpha_i) \cdot v = \sum_{i=1}^r k_i \alpha_i$  in  $\mathbf{S}_v$ , a contradiction as the paths  $\alpha_i$  are  $K$ -independent. Thus each  $x_r$  is a  $K$ -linear combination of finitely many monomials of the form  $\alpha_i \beta_i^*$ . So  $N = \sum_{i=1}^n L \alpha_i \beta_i^*$ , where  $n$  is some positive integer and for each  $i = 1, \dots, n$ ,  $s(\beta_i) = v$  and  $r(\alpha_i) = r(\beta_i)$ . Since  $v$  is an infinite emitter, we can choose an edge  $f$  with  $s(f) = v$  which is not the initial edge of any of the paths  $\beta_i$  and so  $\beta_i^* f = 0$  for  $i = 1, \dots, n$ . Now  $ff^* \in N$  (as  $ff^* \cdot v = 0$ ) and so we can write  $ff^* = \sum_{i=1}^n b_i \alpha_i \beta_i^*$  where  $b_i \in L$ . But then  $ff^* = ff^* ff^* = \sum_{i=1}^n b_i \alpha_i \beta_i^* ff^* = 0$ , a contradiction. Hence  $N$  is not finitely generated and consequently  $\mathbf{S}_v$  is not finitely presented. ■

**Corollary 3.3** *Let  $E$  be an arbitrary graph. If every simple left module over  $L_K(E)$  is finitely presented, then  $E$  must be a row-finite graph.*

In view of Corollary 3.3, we assume hereafter that  $E$  is a row-finite graph.

**Remark 3.4** *Let  $p$  be an infinite path. If the corresponding Chen simple module  $V_{[p]}$  is projective, then it is clearly finitely presented. In this case, Chen [17] showed that path  $p$  will be tail equivalent to the infinite line segment  $\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \dots$ . Thus  $V_{[p]}$  will be projective if and only if the path  $p$  contains a line point.*

It was shown in (Proposition 4.1, [12]) that if  $E$  is a finite graph, then for any infinite irrational path  $p$ , the Chen simple module  $V_{[p]}$  is not finitely presented. We wish to generalize this result to the case when  $E$  is a row-finite graph. Specifically we show that if  $p$  is an infinite irrational path in a row-finite graph  $E$  such that no vertex on  $p$  is a line point, then the Chen simple module  $V_{[p]}$  is not finitely presented. To accomplish this, we fix some notation and make some initial useful remarks.

Let  $E$  be a row-finite graph and let  $p = e_1 e_2 \cdots e_n \cdots$  be an infinite irrational path in  $E$  such that no vertex on the path  $p$  is a line point. This means that there is an infinite sequence of integers  $n_1 < \cdots < n_i < n_{i+1} \cdots$  such that, for each  $i$ , there is a bifurcation at  $s(e_{n_i}) = v_{n_i}$ . For convenience, we shall call the  $n_i$  *bifurcating integers*. Let  $v = s(e_1) = s(p)$ . Writing the simple module  $V_{[p]} = L \cdot p$ , we have an exact sequence of  $L$ -modules

$$0 \longrightarrow N \longrightarrow Lv \xrightarrow{\theta} V_{[p]} \longrightarrow 0 \quad (*)$$

where  $\theta(a) = a \cdot p$  for all  $a \in Lv$ . Observe that if  $f$  is a bifurcating edge with  $s(f) = v_{n_i}$ , then  $\theta(f^* e_{n_i-1}^* \cdots e_1^*) = 0$  and so, for each  $i$ , the left ideal

$$L_{n_i} = \sum_{f \in s^{-1}(v_{n_i}), f \neq e_{n_i}} L f^* e_{n_i-1}^* \cdots e_1^* \subseteq N.$$

It is an easy argument to show that  $\sum_{i \geq 1} L_{n_i} = \bigoplus_{i \geq 1} L_{n_i}$ . Indeed, suppose

$$a_{n_1} + \cdots + a_{n_k} = 0 \quad (\#)$$

where  $a_{n_i} \in L_{n_i}$ . Denoting  $\mu_{n_k} = e_1 \cdots e_{n_k-1} e_{n_k-1}^* \cdots e_1^*$ , observe that, for any  $i = 1, \dots, k$ ,  $a_{n_i} \mu_{n_k} = a_{n_k}$  or 0 according as  $i = k$  or not. Then multiplying the equation  $(\#)$  on the right by  $\mu_{n_k}$ , we get  $a_{n_k} = 0$ . Proceeding like this, we obtain that  $a_{n_i} = 0$  for all  $i = 1, \dots, k$ , thus showing that our sum is direct. So,  $\bigoplus_{i \geq 1} L_{n_i} \subseteq N$ .

To establish that  $V_{[p]}$  is not finitely presented, all we need is to show that

$$N = \bigoplus_{i=1}^{\infty} L_{n_i}. \quad (\#\#)$$

However, in [4], the authors investigate projective resolutions of simple  $L_K(E)$ -modules and, in doing so, they proceed as we have done to consider a simple



module  $V_{[p]}$  induced by infinite irrational path  $p$  not containing a line point and indeed show, in our notation, that  $N = \bigoplus_{i=1}^{\infty} L_{n_i}$  (see Lemmas 2.14 and 2.15 and Corollary 2.6 in [4]). Since our proof of  $(\#\#)$  is essentially the same as given in [4], we omit the proof and refer to [4] for a justification of  $(\#\#)$ . A consequence of  $(\#\#)$  is the following.

**Corollary 3.5** *Let  $E$  be a row-finite graph. If  $p$  is an infinite irrational path in  $E$  containing no line points, then the Chen simple module  $V_{[p]}$  is not finitely presented.*

Next, we recall a pre-order  $\geq$  that was introduced in [12] (see also [7]) among the cycles in the graph  $E$ .

**Definition 3.6** ([12]) *Given two cycles  $c, c'$  in  $E$ , define  $c \geq c'$  if there is a path from a vertex on  $c$  to a vertex on  $c'$ .*

As a consequence of Corollary 3.3 and Corollary 3.5, we derive the following Proposition which summarises the necessary conditions on the graph  $E$  in order that every simple left  $L$ -module over  $L = L_K(E)$  is finitely generated.

**Proposition 3.7** *Let  $E$  be an arbitrary graph that contains cycles. If every simple left  $L$ -module is finitely presented, then  $E$  satisfies the following properties:*

- (i)  $E$  is row-finite;
- (ii) Distinct cycles in  $E$  have no common vertex;
- (iii) The set  $C$  of all cycles in  $E$  is a non-empty artinian partially ordered set under the relation  $\geq$ ;
- (iv) Every infinite path in  $E$  either contains a line point or is tail equivalent to a rational path.

**Proof.** (i) Follows from Corollary 3.3.

(ii) If the same vertex  $v$  is the base of two different cycles  $g, h$ , then we get an infinite irrational path  $p = ghg^2h^2 \cdots g^nh^n \cdots$ . Then Corollary 3.5 implies that the simple module  $V_{[p]}$  is not finitely presented, a contradiction.

(iii) Now Condition (ii) implies that the relation  $\geq$  is anti-symmetric. Thus  $\geq$  is a partial order. If there is an infinite descending chain of distinct cycles in  $C$ , then this chain can be expanded to an infinite irrational path  $p$  in  $E$ . This leads to a contradiction since the corresponding simple module  $V_{[p]}$  is, by Corollary 3.5, not finitely presented. Thus  $(C, \geq)$  is an artinian partially ordered set.

(iv) Corollary 3.5 implies that if an infinite path in  $E$  is not rational, then it must contain a line point. ■

In preparation for proving our main theorem, we recall the following Definition of a "hedgehog" graph from [10].

**Definition 3.8** Suppose  $E$  is a row-finite graph and  $H$  is a non-empty hereditary saturated subset of vertices in  $E$ .

Let  $F(H) = \{\text{paths } \alpha = e_1 \cdots e_n : n \geq 1, r(e_i) \notin H \text{ for } i = 1, \dots, n-1 \text{ and } r(e_n) \in H\}$ .

Let  $\bar{F}(H) = \{\bar{\alpha} : \alpha \in F(H)\}$ .

Then the "hedgehog" graph  ${}_H E = ({}_H E^0, {}_H E^1, s', r')$  is defined as follows:

- (1)  ${}_H E^0 = H \cup F(H)$ .
- (2)  ${}_H E^1 = \{e \in E^1 : s(e) \in H\} \cup \bar{F}(H)$ .
- (3) For every  $e \in E^1$  with  $s(e) \in H$ ,  $s'(e) = s(e)$  and  $r'(e) = r(e)$ .
- (4) For every  $\bar{\alpha} \in \bar{F}(H)$ ,  $s'(\bar{\alpha}) = \alpha$  and  $r'(\bar{\alpha}) = r(\alpha)$ .

It was shown in (Lemma 1.2, [10]) that, for a row-finite graph  $E$ , if  $I$  is a graded ideal of  $L_K(E)$  with  $I \cap E^0 = H$ , then  $I \cong L_K({}_H E)$ . Thus, in particular, the graded ideal  $I$  is a ring with local units.

Following [3], we call a ring  $R$  with local units *categorically left noetherian* if submodules of finitely generated left  $R$ -modules are again finitely generated. It was shown in [3] that, for any index set  $\Lambda$ , the matrix ring  $M_\Lambda(K[x, x^{-1}])$  is categorically noetherian. Also, as a special case of Theorem 3.9 of [3], one obtains that, for a graph  $E$ ,  $L_K(E) \cong M_\Lambda(K[x, x^{-1}])$  if and only if  $E$  contains a unique cycle  $c$  without exits,  $T_E(v) \cap c^0 \neq \emptyset$  for every vertex  $v$ , and every infinite path in  $E$  is tail-equivalent to the infinite rational path  $ccc \cdots$ .

The next Proposition plays a key role in proving our main theorem.

**Proposition 3.9** Suppose  $E$  is a row-finite graph and  $M$  is a graded ideal of  $L = L_K(E)$  such that  $L/M \cong M_\Lambda K[x, x^{-1}]$  where  $\Lambda$  is an arbitrary index set. If every simple left  $M$ -module is finitely presented, then every simple left  $L$ -module is finitely presented.

**Proof.** Let  $S$  be a simple left  $L$ -module.

Case 1: Suppose  $MS = S$ . Then  $S$  is a simple  $M$ -module and, by hypothesis, is finitely presented as an  $M$ -module. Let  $H = M \cap E^0$ . Since  $M$  is a graded ideal,  $M \cong L_K({}_H E) = L'$  ([10]). By Lemma 3.1,  $S \cong L'u/A$  for some vertex  $u \in ({}_H E)^0$  where  $A$  is a finitely generated maximal  $L'$ -submodule of  $L'u$ . Under the isomorphism  $L' \cong M$ , let  $u$  map to an idempotent  $\epsilon$  in  $M$  and  $A$  map to a submodule  $B$  of  $M$ . Since  $M$  has local units,  $L\epsilon = M\epsilon$ ,  $B$  is a maximal  $L$ -submodule of  $L\epsilon$  and  $S \cong L\epsilon/B$ . As  $L$  is projective as a left  $L$ -module (see [15]),  $L\epsilon$  is a cyclic projective summand of  $L$  and  $B$  is a finitely generated  $L$ -submodule. Hence  $S \cong L\epsilon/B$  is finitely presented as a left  $L$ -module.

Case 2: Suppose  $MS = 0$  so  $S \cong L/Y$  for some maximal left ideal  $Y \supseteq M$ . From Lemma 3.1, it is clear that there is a vertex  $v \notin Y$  such that  $Y = (Lv \cap Y) \oplus L(1-v)$  and  $S \cong Lv/N$ , where  $N = Lv \cap Y$ . If  $v$  is a sink, then  $Lv$  will be simple and a direct summand of  $L$  ([14]) and so  $S \cong Lv$  is projective and finitely presented. Suppose  $v$  is not a sink. Since  $M$  is a two-sided ideal,  $M = (Lv \cap M) \oplus (L(1-v) \cap M)$  and clearly,  $(Lv \cap M) \subset N = Lv \cap Y$ . Let  $H = M \cap E^0$ . Now  $L_K(E \setminus H) \cong L/M \cong M_\Lambda(K[x, x^{-1}])$  for some index set  $\Lambda$ . This means, by Theorem 3.9 of [3] (see also Theorem 4.2.12, [2]) that

$E \setminus H$  has a unique cycle  $c$  without exits based at a vertex  $u$ ,  $T_E(v) \cap c^0 \neq \emptyset$  for every vertex  $v$  and every infinite path in  $E \setminus H$  is tail-equivalent to the rational path  $ccc \cdots$ . Now  $(N + M)/M$  is a submodule of the cyclic submodule  $(Lv + M)/M$  and, since  $L/M \cong M_\Lambda(K[x, x^{-1}])$  is categorically noetherian (see Lemma 1.3, [3]),  $N/(Lv \cap M) \cong (N + M)/M$  is finitely generated. Thus  $N = Lx_1 + \cdots + Lx_r + (Lv \cap M)$  where the  $x_i \in Lv$ . So to prove that  $S$  is finitely presented, we need only to show that  $Lv \cap M$  is finitely generated. To start with, we claim that  $T_{E \setminus H}(v)$  is a finite set. To justify this, we follow an argument that is embedded in the proof of Proposition 3.6 of [3]. Suppose, on the contrary,  $T_{E \setminus H}(v)$  is an infinite set. Then  $v \notin c^0$ , as  $c^0$  is a hereditary set due to  $c$  having no exits and is finite. Since  $v$  is a regular vertex, there is then an edge  $e_1$  with  $s(e_1) = v$ ,  $r(e_1) = v_1$  such that  $T_{E \setminus H}(v_1)$  is an infinite set. Clearly,  $v_1 \notin c^0$  and  $v_1$  is a regular vertex. Repeating this process, we obtain an infinite path in  $E \setminus H$  such that no vertex on this path lies on  $c$ . This contradicts the fact the every infinite path in  $E \setminus H$  is tail-equivalent to the rational path  $ccc \cdots$ . Thus  $T_{E \setminus H}(v)$  is a finite set. We now distinguish two cases.

Case A: Suppose  $v \notin c^0$ . Since  $E \setminus H$  is row-finite, since every infinite path in  $E \setminus H$  is tail-equivalent to the rational path  $ccc \cdots$  and since  $T_{E \setminus H}(v)$  is a finite set, the number of paths  $\alpha$  in  $E \setminus H$  satisfying  $s(\alpha) = v$  and  $r(\alpha) \notin c^0$  is finite. Among these finitely many paths, let  $\gamma_1, \dots, \gamma_m$  be the listing of all those paths with the property that  $s(\gamma_j) = v$  and  $r(\gamma_j) = u_j$  such that there is at least one  $e \in s^{-1}(u_j)$  with  $r(e) \in H$ . Here we use the convention that  $u_j = v$  if  $\gamma_j$  has length 0. For each  $j = 1, \dots, m$ , let  $\{e_{jk} : k = 1, \dots, l_j\}$  be the set of all the edges  $e_{jk} \in s^{-1}(u_j)$  such that  $r(e_{jk}) \in H$ . Now each element of  $Lv \cap M$  is a  $K$ -linear combination of monomials of the form  $pq^*$  where  $s(q) = v$  and  $r(q) \in H$ . It is then clear that, each such path  $q$  is of the form  $q = \gamma_j e_{jk} q'$  for some  $j$  and  $k$ , where  $q'$  is a suitable path. This means that  $pq^* \in Le_{jk}^* \gamma_j^*$ . As  $e_{jk}^* \gamma_j^* \in Lv \cap M$ , for all  $j, k$ , we then conclude that

$$Lv \cap M = \sum_{j=1}^m \sum_{k=1}^{l_j} Le_{jk}^* \gamma_j^*.$$

Consequently,  $N = \sum_{i=1}^r Lx_i + (Lv \cap M)$  is finitely generated. This shows that the simple module  $S = Lv/N$  is finitely presented.

Case B: Suppose  $v \in c^0$ . In this case  $Lv \cap M = \{0\}$ , as there are no paths connecting  $v$  to a vertex in  $H$ . Then  $N \cong N/(Lv \cap M)$  is finitely generated and so  $Lv/N$  is finitely presented. ■

We are now ready to prove the main theorem of this section.

**Theorem 3.10** *Suppose  $E$  is an arbitrary graph that contains cycles and  $L = L_K(E)$ . Then the following conditions are equivalent:*

- (1) *Every simple left  $L$ -module is finitely presented;*
- (2) *(a)  $E$  is row-finite, (b) the set  $C$  of all the cycles in  $E$  is a non-empty artinian partially ordered set under the defined relation  $\geq$ , (c) every infinite*

path in  $E$  either contains a line point or is tail equivalent to a rational path and  
(d) for every proper hereditary saturated set  $H$  of vertices containing all the line points,  $E \setminus H$  contains cycles without exits but no line points;

(3)  $L$  is the union of a smooth well-ordered ascending chain of graded ideals

$$0 \leq I_1 < \cdots < I_\alpha < I_{\alpha+1} < \cdots \quad (\alpha < \tau) \quad (***)$$

where  $\tau$  is a suitable ordinal,  $I_1 = \text{Soc}(L)$  (which may be 0) and, for each  $\alpha \geq 1$ ,  $I_{\alpha+1}/I_\alpha \cong M_{\Lambda_\alpha}(K[x, x^{-1}])$  where  $\Lambda_\alpha$  is an index set that depends on  $\alpha$ .

**Proof.** Assume (1). We need only to prove Condition 2(d), as Conditions 2(a) - 2(c) follow from Proposition 3.7. Let  $H \neq E^0$  be a hereditary saturated subset of vertices containing all the line points in  $E$ . First we show that  $E \setminus H$  contains no line points. Suppose, by way of contradiction,  $v$  is a line point in  $(E \setminus H)^0 = E^0 \setminus H$ . Since  $E$  is row-finite and  $H$  contains all the sinks in  $E$ ,  $v$  is not a sink in  $E^0 \setminus H$ . So  $T_{E \setminus H}(v)$  consists of the infinite set  $\{v = v_1, v_2, \dots, v_n, \dots\}$  of vertices having no bifurcations in  $E \setminus H$  and they define an infinite path

$$\bullet \longrightarrow \bullet \cdots \bullet \longrightarrow \cdots$$

$v_1 \qquad v_2 \qquad v_n$

First note that, in  $E$ , none of these  $v_i$  can be a base of a cycle  $c$ , because otherwise  $v_i$  will be a base of  $c$  in  $E \setminus H$ , a contradiction. Also none of the  $v_i$  can be a line point in  $E$  as  $H$  contains all the line points. Thus these vertices  $v_i$  define an infinite irrational path in  $E$  not containing any line points. This is impossible in view of Corollary 3.5. Hence  $E \setminus H$  contains no line points. As every simple left module over  $L_K(E \setminus H) \cong L/I(H)$  is finitely presented, Theorem 2.2 (iii) then implies that  $E \setminus H$  must contain a cycle. Moreover, by Proposition 3.7, the cycles in  $E \setminus H$  form a non-empty artinian partially ordered set  $C$  under the relation  $\geq$ . Since  $E \setminus H$  does not contain any line points, Proposition 3.7(iv) implies that any minimal element in  $C$  will be a cycle without exits.

Assume (2). Let  $I_1$  be the ideal generated by all the line points in  $E$ . Then  $I_1$  is a graded ideal and is the socle of  $L$  [14]. Note that  $I_1$  may be zero, but  $I_1 \neq L$  since  $E$  contains cycles. Suppose  $\alpha > 1$  and that the graded ideals  $I_\beta$  have been defined for all  $\beta < \alpha$  with the stated properties. If  $\alpha$  is a limit ordinal, define  $I_\alpha = \cup_{\beta < \alpha} I_\beta$ . Suppose  $\alpha = \beta + 1$  and that  $I_\beta \neq L$ . Let  $H = I_\beta \cap E^0$ . Now  $E \setminus H$  satisfies the Conditions (2)(a) - (d) and, in particular,  $E \setminus H$  contains cycles without exits. Since  $L_K(E \setminus H) \cong L/I_\beta$ , identifying  $L/I_\beta$  with  $L_K(E \setminus H)$ , define  $I_{\beta+1}/I_\beta$  to be the (graded) ideal generated by the vertices in a single cycle without exits in  $E \setminus H$ . By (Proposition 3.7(iii), [3]),  $I_{\beta+1}/I_\beta$  is isomorphic to a matrix ring of the form  $M_{\Lambda_\beta}(K[x, x^{-1}])$  where  $\Lambda_\beta$  is an arbitrary index set. Proceeding like this and applying transfinite induction, we obtain the transfinite chain (\*\*\*) of graded ideals, where the successive quotients are matrix rings of appropriate size over  $K[x, x^{-1}]$ . This proves (3).

Assume (3). We are given  $L$  is the union of the transfinite chain (\*\*\*) of graded ideals  $I_\alpha$  with the stated properties. First of all observe that, by Lemma 3.1, every simple left  $L$ -module is isomorphic to  $Lv/N$  for some vertex  $v$  in  $E$ . Now the vertex  $v$  in  $E$  belongs to some graded ideal  $I_\alpha$  and each  $I_\alpha$  is a ring

with local units as  $I_\alpha \cong L_{K(H_\alpha)} E$  where  $H_\alpha = I_\alpha \cap E^0$ . This means that the  $L$ -submodules of  $I_\alpha$  coincide with the  $I_\alpha$ -submodules of  $I_\alpha$ . Consequently, every simple left  $L$ -module is a simple left  $I_\alpha$ -module for suitable  $\alpha$ . So we wish to show, by transfinite induction on  $\alpha$ , that every simple left  $I_\alpha$ -module is finitely presented as a simple left  $L$ -module. If  $\alpha = 1$ , this is immediate since  $I_1$ , being the socle  $L$ , is a direct sum of projective simple left ideals of  $L$ .

Assume  $\alpha \geq 2$  and that, for all  $\beta < \alpha$ , every simple left  $I_\beta$ -module is a finitely presented simple left  $L$ -module. Suppose  $\alpha$  is a limit ordinal so that  $I_\alpha = \bigcup_{\beta < \alpha} I_\beta$ . Since  $I_\alpha$  is a graded ideal,  $I_\alpha \cong L_{K(H_\alpha)} E$  where  $H_\alpha = I_\alpha \cap E^0$ .

Now any simple left  $I_\alpha$ -module  $S$  is of the form  $I_\alpha v / N$  where  $v \in (H_\alpha E)^0$ . Note that  $v \in H_\alpha$  or  $v \in F(H_\alpha)$  (see Definition 3.8). In either case, since  $v \in I_\beta$  for some  $\beta < \alpha$ ,  $S$  becomes a simple left  $I_\beta$ -module and so, by induction, is a finitely presented simple left  $L$ -module. Suppose  $\alpha = \beta + 1$  for some  $\beta \geq 1$ . As before the graded ideal  $I_\alpha \cong L_{K(H_\alpha)} E$  where  $H_\alpha = I_\alpha \cap E^0$ . Let  $S$  be a simple left  $I_\alpha$ -module. Since  $I_{\beta+1}/I_\beta \cong M_{\Lambda_\alpha}(K[x, x^1])$  for some index set  $\Lambda_\alpha$  and since every simple  $I_\beta$ -module is finitely presented as an  $I_\beta$ -module (also as an  $L$ -module), we appeal to Proposition 3.9 to conclude that  $S$  is finitely presented as a simple left  $I_\alpha$ -module and hence is a finitely presented simple left  $L$ -module. Applying transfinite induction, we reach the desired conclusion. This proves (1). ■

When  $E$  is a finite graph, Conditions 2(a) and 2(c) of the above theorem are immediate. Condition 2(d) is also automatically satisfied. To see this, suppose  $H$  is a proper hereditary saturated subset of vertices in the finite graph  $E$  containing all the line points. Since every vertex in  $E$  is regular,  $E \setminus H$  contains no sinks (and no line points). This means (since  $E$  is finite) that every path in  $E \setminus H$  eventually ends at a cycle. In particular, Condition 2(c) holds. Also it is easy to see that the condition that  $\geq$  is antisymmetric (as part of Condition 2(b)) is equivalent to stating that different cycles in  $E$  have no common vertex. Thus we derive the following characterization, proved in [12], of a Leavitt path algebra  $L$  of a finite graph  $E$  over which every simple left  $L$ -module is finitely presented.

**Corollary 3.11** ([12]) *Let  $E$  be any finite graph,  $K$  be any field and let  $L = L_K(E)$ . Then every simple left  $L$ -module is finitely presented if and only if every vertex in  $E$  is the base of at most one cycle.*

EXAMPLE: As an example of a graph satisfying the conditions of Theorem 3.10, consider a graph  $E'$  consisting of infinitely many loops  $c_i$  based at vertices  $v_i$  for  $i = 1, 2, \dots$  such that, for each  $i$ , there is an edge  $e_i$  with  $s(e_i) = v_{i+1}$  and  $r(e_i) = v_i$ . In addition, there is a vertex  $w$  and an edge  $e$  with  $s(e) = v_1$  and  $r(e) = w$ . Thus  $w$  is a sink and the only line point in  $E$ . Let  $H_0 = \{w\}$  and, for each  $n \geq 1$ , let  $H_n = \{w, v_1, \dots, v_n\}$ . Clearly, the proper non-empty hereditary saturated subsets of  $(E')^0$  are just the sets  $H_n$ ,  $n \geq 0$ . For each  $n \geq 0$ , the quotient graph  $E' \setminus H_n$  contains a cycle without exits and has no line points. Moreover the cycles  $c_i$  in  $E'$  form an artinian partially ordered set under the relation  $\geq$ .

It is now clear that Theorem 1.1 follows from Theorems 2.2 and 3.10.

## 4 A Corner-Tree Phenomena

This section contains some preliminary results which will be used in the next section. We explore the conditions needed for the corner  $vLv$ , where  $v$  is vertex, of a Leavitt path algebra  $L = L_K(E)$  to have a various ring properties. These seem to be governed by appropriate graph properties of the tree  $T_E(v)$ . Our focus is when  $v$  is an acyclic vertex (see definition below).

In the following, we make the convention that if  $u, w \in T_E(v)$ , then  $T_E(v)$  contains all the edges in the paths connecting  $u$  to  $w$ . Thus  $T_E(v)$  is a complete subgraph of  $E$ .

We shall be using the following generalization of the "hedgehog" graph given in Definition 3.8 to arbitrary graphs (see [26]).

**Result (a).** Let  $E$  be an arbitrary graph and let  $I$  a graded ideal of  $L_K(E)$  with  $I \cap E^0 = H$  and  $S = \{v \in B_H : v^H \in I\}$ . Then Theorem 6.1 of [26] shows that  $I = I(H, S)$  is isomorphic to a Leavitt path algebra  $L_K(\bar{E}(H, S))$  where the graph  $\bar{E}(H, S)$  is defined as follows:

Let  $F_1 = \{\text{paths } \alpha = e_1 \cdots e_n : n \geq 1, r(e_i) \notin H \text{ for } i = 1, \dots, n-1 \text{ and } r(e_n) \in H\}$ .

Let  $F_2 = \{\text{paths } \alpha : r(\alpha) \in S \text{ and length of } \alpha \geq 1\}$ .

For  $i = 1, 2$ , let  $\bar{F}_i = \{\bar{\alpha} : \alpha \in F_i\}$ .

Then  $(\bar{E}(H, S))^0 := H \cup S \cup F_1 \cup F_2$ ;

$(\bar{E}(H, S))^1 := \{e \in E^1 : s(e) \in H\} \cup \{e \in E^1 : s(e) \in S, r(e) \in H\} \cup \bar{F}_1 \cup \bar{F}_2$ ,

and  $s, r$  are extended to  $\bar{E}(H, S)$  by setting  $s(\bar{\alpha}) = \alpha$  and  $r(\bar{\alpha}) = r(\alpha)$  for all  $\alpha \in F_1 \cup F_2$ .

Thus every graded ideal of  $L_K(E)$ , being isomorphic to a Leavitt path algebra, is possessed with local units.

We begin with the following result from [18].

**Lemma 4.1** ([18]) *Let  $\bar{H}$  be the saturated closure of a hereditary set  $H$  of vertices. If a vertex  $w \in \bar{H}$  and  $w$  lies on a closed path, then  $w \in H$ .*

As a consequence of the above Lemma, we obtain the following.

**Lemma 4.2** *Let  $E$  be an arbitrary graph and  $v \in E^0$ . If the ideal  $I$  generated by  $v$  contains a closed path  $c$  then  $c^0 \subset T_E(v)$ .*

**Proof.** This follows immediately if one observes that  $T_E(v)$  is a hereditary set and its saturated closure is  $I \cap E^0$ . ■

It is known (see [14]) that a vertex  $v$  is a line point (i.e.,  $T_E(v)$  is a single straight line segment) if and only if the corner  $vLv \cong K$ . Likewise, by examining the representation of elements in  $vLv$ , it is clear that  $T_E(v)$  is a cycle without exits if and only if the corner  $vLv \cong K[x, x^{-1}]$ . We explore below similar connections between  $T_E(v)$  and  $vLv$ .

**Proposition 4.3** *Let  $E$  be an arbitrary graph. Then the following are equivalent for any vertex  $v$  in  $E$ :*

- (i) *The corner  $vLv$  is von Neumann regular;*
- (ii)  *$T_E(v)$  contains no cycles;*
- (iii) *The ideal  $I$  generated by  $v$  is von Neumann regular.*

**Proof.** (i)  $\Rightarrow$  (ii). Assume  $vLv$  is von Neumann regular. Suppose, by way of contradiction,  $T_E(v)$  contains a cycle  $c$ . Let  $p$  be a path from  $v$  to a vertex  $w$  on  $c$ . Let  $\gamma$  denote  $v + pc p^* \in vLv$ . We wish to show that there is no  $a = vav \in vLv$  such that  $\gamma a \gamma = \gamma$ . Suppose, by way of contradiction, such an  $a$  exists. Write  $a = \sum_{i=r}^s a_i$  as a graded sum of homogeneous elements where  $r, s \in \mathbb{Z}$  with  $r \leq s$ . Substituting for  $\gamma$  and  $a$  in the equation  $\gamma a \gamma = \gamma$ , we obtain the equation

$$(v + pc p^*) \sum_{i=r}^s a_i (v + pc p^*) = (v + pc p^*).$$

To reach a contradiction, we essentially follow the ideas in the proof of (1)  $\Rightarrow$  (2) in Theorem 1 of [5] by expanding the above equation, considering it as a graded equation and comparing the degree of the components on both sides. Existence of terms of higher degrees on the left hand side with no terms of equal degree on the right hand side leads to a contradiction. Hence  $T_E(v)$  contains no cycles, thus proving (ii).

(ii)  $\Rightarrow$  (iii). If  $T_E(v)$  contains no cycles, then by Lemma 4.2  $H = I \cap E^0$  contains no cycles. Now, by Result (a), the graded ideal  $I \cong L_K(\bar{E}(H, \emptyset))$ . From its definition it is clear that the graph  $\bar{E}(H, \emptyset)$  also contains no cycles. Then we appeal to Theorem 1 of [5] to conclude that  $I \cong L_K(\bar{E}(H, \emptyset))$  is von Neumann regular, thus proving (iii).

(iii)  $\Rightarrow$  (i). If  $I$  is von Neumann regular, then so is the corner  $vLv = vIv$ .

■

For the convenience of later use, we introduce the following definition.

**Definition 4.4** *A vertex  $v$  in a graph  $E$  is called an **acyclic vertex** if it satisfies the equivalent conditions of Proposition 4.3.*

As an immediate consequence of Proposition 4.3, we get the following result.

**Corollary 4.5** *Let  $E$  be an arbitrary graph and let  $A$  be the ideal generated by the set of all the acyclic vertices in  $E$ . Then  $A$  is von Neumann regular.*

**REMARK: As an aside,** we wish to point out that Proposition 4.3 appears to be one instance of the phenomenon that when a graph property  $P$  of the graph  $E$  is equivalent to a ring property  $Q$  of the ring  $L = L_K(E)$ , then for any vertex  $v \in E$ ,  $T_E(v)$  has the property  $P \iff vLv$  has the property  $Q$ .

We digress a bit to point two other instances of such a phenomenaon. Two special graph properties of  $E$  which play an important role in the investigation of  $L_K(E)$  are Condition (L) and Condition (K). It was shown in [25] that  $E$

satisfies Condition (L) if and only if the Leavitt path algebra  $L = L_K(E)$  is a Zorn ring. Here, a ring  $R$  is a *Zorn ring* if given any  $a \in R$ , there is a  $b \in R$  such that  $bab = b$  (For other equivalent definitions, see [25]). Likewise, it was shown in [15] that the graph  $E$  satisfies Condition (K) if and only if  $L_K(E)$  is right/left weakly regular. Recall that a ring  $R$  is right *weakly regular* if to each  $a \in R$  there is an  $x \in RaR$  such that  $a = ax$ . (see [15] for other equivalent definitions and properties of left/right weakly regular rings). In general, weak regularity and being a Zorn ring are independent properties, neither implying the other (see [25] for examples).

**Proposition 4.6** *Let  $E$  be an arbitrary graph and  $L = L_K(E)$ . Then, for a vertex  $v \in E$ , the following are equivalent:*

- (i) *The corner  $vLv$  is a Zorn ring;*
- (ii)  *$T_E(v)$  satisfies Condition (L);*
- (iii) *The ideal  $I$  generated by  $v$  is a Zorn ring.*

**Proof.** (i)  $\Rightarrow$  (ii). Assume  $vLv$  is a Zorn ring. Suppose, on the contrary,  $T_E(v)$  contains a cycle  $c$  without exits in  $T_E(v)$  and hence in  $E$ . Let  $p$  be a path from  $v$  to a vertex  $w$  on  $c$ . Then the map  $\theta : wLw \rightarrow vLv$  given by  $\theta(x) = pxp^*$  is a monomorphism. Now  $wLw \cong K[x, x^{-1}]$  as  $c$  has no exits. So, for  $\theta(w) = \epsilon$  we have  $\epsilon L \epsilon \cong wLw \cong K[x, x^{-1}]$  and this is a contradiction since, being a corner of the Zorn ring  $vLv$ ,  $\epsilon L \epsilon = \epsilon vLv \epsilon$  must be a Zorn ring (see [25]), but the integral domain  $K[x, x^{-1}]$  is obviously not a Zorn ring. Hence every cycle in  $T_E(v)$  must have an exit, thus proving (ii).

(ii)  $\Rightarrow$  (iii). The proof is similar to that of (i)  $\Rightarrow$  (ii) of Proposition 4.3. By Lemma 4.2,  $T_E(v)$  satisfies Condition (L) exactly when Condition (L) holds in  $H = I \cap E^0$ . Then the graph  $\bar{E}(H, \emptyset)$  also satisfies Condition (L). Consequently, by Theorem 2.1 of [25],  $I \cong L_K(\bar{E}(H, \emptyset))$  is a Zorn ring.

(iii)  $\Rightarrow$  (i). If  $I$  is a Zorn ring, then so is the corner  $vLv = vLv$  (see [25]).

■

**Proposition 4.7** *Let  $E$  be an arbitrary graph. Then the following are equivalent for any vertex  $v$  in  $E$ :*

- (i) *The corner  $vLv$  is left/right weakly regular and a Zorn ring;*
- (ii)  *$T_E(v)$  satisfies Condition (K);*
- (iii) *The ideal  $I$  generated by  $v$  is both left/right weakly regular and a Zorn ring.*

**Proof.** (i)  $\Rightarrow$  (ii). Suppose  $vLv$  is both left/right weakly regular and a Zorn ring. We wish to show  $T_E(v)$  satisfies Condition (K). Already by Proposition 4.6,  $T_E(v)$  satisfies Condition (L). The proof that Condition (K) holds follows the ideas in some earlier papers (see, for e.g., Proposition 3.9 in [15]). Suppose Condition (K) does not hold in  $T_E(v)$ . Then there is a vertex  $w \in T_E(v)$  which is the base of only one closed path (cycle)  $c$ . Now  $c$  has exits in  $T_E(v)$  due to Condition (L). Let  $X$  denote the hereditary closure of the set  $\{r(e) : e \text{ an exit for } c\}$  and  $Y$  be the saturated closure of  $X$ . Since  $T_E(v)$  does not satisfy Condition (K), Lemma 4.1 implies that  $Y \cap c^0 = \emptyset$ . If  $J = I(Y, \emptyset)$  is the



(graded) ideal generated by  $Y$ , then  $L/J \cong L_K(E \setminus (Y, \emptyset))$  and in  $E \setminus (Y, \emptyset)$ ,  $c$  is a cycle without exits based at  $w$ . For convenience, denote  $L_K(E \setminus (Y, \emptyset))$  by  $\bar{L}$ . Let  $p$  be a path connecting  $v$  to  $w$  in  $E \setminus (Y, \emptyset)$ . Then, as was done in the proof of Proposition 4.6, the map  $\theta : w\bar{L}w \rightarrow v\bar{L}v$  given by  $\theta(x) = p x p^*$  is a monomorphism. Now  $w\bar{L}w \cong K[x, x^{-1}]$  as  $c$  has no exits. If  $\epsilon = \theta(w)$ , then  $K[x, x^{-1}] \cong \epsilon\bar{L}\epsilon = \epsilon v\bar{L}v\epsilon$ , a corner of  $v\bar{L}v$ . This is a contradiction, since  $v\bar{L}v$ , being a homomorphic image of  $vLv$ , is left/right weakly regular while the commutative integral domain  $K[x, x^{-1}]$  is not weakly regular. Hence  $T_E(v)$  satisfies Condition (K), thus proving (ii).

(ii)  $\Rightarrow$  (iii) Again the proof is similar to the proof of (i)  $\Rightarrow$  (ii) in Proposition 4.3. If  $T_E(v)$  satisfies Condition (K), then by Lemma 4.2, Condition (K) holds in  $H = I \cap E^0$  and also in the graph  $\bar{E}(H, \emptyset)$ . Then, by [15],  $I \cong L_K(\bar{E}(H, \emptyset))$  is left/right weakly regular. Since Condition (K) implies Condition (L), we appeal to Theorem 2.1 of [25] to conclude that  $L_K(\bar{E}(H, \emptyset))$  is also a Zorn ring. This proves (ii).

(iii)  $\Rightarrow$  (i). This immediate from the fact that a corner of a left/right weakly regular Zorn ring is again left/right weakly regular Zorn ring. ■

## 5 Leavitt path algebras with finite Gelfand-Kirillov dimension

As we noted Corollary 3.11, when  $E$  is a finite graph all the irreducible representations of  $L_K(E)$  are finitely presented exactly when distinct cycles in  $E$  are disjoint. Interestingly, it was shown in [7] that this same condition for a finite graph  $E$  is equivalent to the Leavitt path algebra  $L_K(E)$  having finite Gelfand-Kirillov dimension. Examples show that this equivalence no longer holds if  $E$  is an infinite graph. In this section, we extend the results of [7], [8] to obtain a complete characterization of and a structure theorem for Leavitt path algebras over an arbitrary graph having finite Gelfand-Kirillov dimension. It turns out that the "building blocks" for these algebras  $L$  are von Neumann regular rings and matrix rings over the Laurent polynomial ring  $K[x, x^{-1}]$ . The acyclic vertices introduced in the previous section play a useful role.

We shall first recall the definition of the Gelfand-Kirillov dimension of associative algebras over a field.

Let  $A$  be a finitely generated algebra over a field  $K$ , generated by a finite dimensional subspace  $V = Ka_1 \oplus \cdots \oplus Ka_m$ . Let  $V^0 = K$  and, for each  $n \geq 1$ , let  $V^n$  denote the  $K$ -subspace of  $A$  spanned by all the monomials of length  $n$  in  $a_1, \dots, a_m$ . Set  $V_n = \sum_{i=0}^n V^i$ . Then the **Gelfand-Kirillov dimension** of  $A$  (for short, the **GK-dimension** of  $A$ ) is defined by

$$\text{GK-dim}(A) := \limsup_{n \rightarrow \infty} \log_n(\dim V_n).$$

It is known that the  $\text{GK-dim}(A)$  is independent of the choice of the generating subspace  $V$ .

If  $A$  is an infinitely generated  $K$ -algebra, then the GK-dimension of  $A$  is defined as

$$\text{GK-dim}(A) := \sup_B \text{GK-dim}(B)$$

where  $B$  runs over all the finitely generated  $K$ -subalgebras of  $A$ .

Some useful examples the GK-dimension of algebras (see [21]) are: The GK-dimension the matrix ring  $M_\Lambda(K)$  is 0 and the GK-dimension of the matrix ring  $M_{\Lambda'}(K[x, x^{-1}])$  is 1, where  $\Lambda, \Lambda'$  are arbitrary index sets.

We also note that if an algebra  $A$  has finite GK-dimension, then every subalgebra of  $A$  and every homomorphic image of  $A$  also has finite GK-dimension. But this does not hold for extensions: If  $I$  is an ideal of an algebra  $A$ , it may happen that both  $I$  and  $A/I$  have finite GK-dimension, but  $A$  has infinite GK-dimension. We refer to [21] for these and for other properties and results on the GK-dimension of algebras.

As mentioned in the first paragraph of this section, we shall now give an example of infinite graph  $E$  in which distinct cycles have no common vertex, but neither all the simple  $L_K(E)$ -modules are finitely presented nor the GK-dimension of  $L_K(E)$  is finite.

Example: Let  $E = F \cup G$  be the union of two graphs  $F, G$  together with a connecting edge  $g$ . Specifically,  $F$  is the graph obtained by removing the vertex  $w$  and edge  $e$  from the graph  $E'$  mentioned at the end of Section 3. Thus  $F$  consists of infinitely many loops  $c_i$  based at vertices  $v_i$  for  $i = 1, 2, \dots$  and, for each  $i$ , there is an edge  $e_i$  with  $s(e_i) = v_{i+1}$  and  $r(e_i) = v_i$ . The graph  $G$  is the countable infinite clock, namely,  $G^0 = \{u\} \cup \{w_i : i = 1, 2, \dots\}$  and  $G^1 = \{f_i : i = 1, 2, \dots\}$  such that, for all  $i$ ,  $s(f_i) = u$  and  $r(f_i) = w_i$ . Finally, there is a connecting edge  $g$  with  $s(g) = v_1$  and  $r(g) = u$ . Now clearly distinct cycles/loops in  $E$  have no common vertex. Since  $E$  is not row-finite, Corollary 3.3 implies that not all simple modules over  $L_K(E)$  are finitely presented. Also  $L_K(E)$  does not have finite GK-dimension. To see this, let, for each  $n > 1$ ,  $F_n$  denote the subgraph where  $(F_n)^0 = \{v_1, \dots, v_n\}$  and  $(F_n)^1 = \{e_1, \dots, e_{n-1}\} \cup \{c_1, \dots, c_n\}$ . Then  $F_n$  is a complete subgraph. If  $B_n$  is the subalgebra generated by  $F_n$ , then by Theorem 5 of [7],  $B_n \cong L_K(F_n)$  has GK-dimension  $2n - 1$ . Consequently,  $\text{GK-dim}(L_K(E)) \geq \sup\{\text{GK-dim}(B_n)\}$  is infinite.

If we consider just the graph  $F$ , then by Theorem 3.10 every simple left/right module over  $L_K(F)$  is finitely presented (and distinct cycles in  $E$  are disjoint), but by the preceding arguments  $L_K(F)$  does not have finite GK-dimension. On the other hand, since  $G$  is acyclic,  $L_K(G)$  is a directed union of direct sums of matrix rings over  $K$  (see Theorem 1, [5]) and so has GK-dimension 0, but not every simple left/right module over  $L_K(G)$  is finitely presented, by Theorem 2.2.

Our goal is to give complete description of the Leavitt path algebras over arbitrary graphs having finite GK-dimension. To accomplish that, we shall be using Result (a) from the previous section together with following Result (b).

**Result (b).** The subalgebra construction using a finite set of edges in a graph ([5]): Let  $E$  be an arbitrary graph and let  $F$  be a finite set of edges in  $E$ . Then the graph  $E_F$  is defined by setting

$$(E_F)^0 = F \cup [(r(F) \cap s(F) \cap s(E^1 \setminus F)) \cup (r(F) \setminus s(F))];$$

$$(E_F)^1 = \{(e, f) \in F \times (E_F)^0 : r(e) = s(f)\} \cup$$

$$\{(e, r(e)) : e \in F \text{ with } r(e) \in (r(F) \setminus s(F))\};$$

Here  $r, s$  are defined by  $s(x, y) = x$  and  $r(x, y) = y$  for any  $(x, y) \in (E_F)^1$ .

A graded monomorphism  $\theta : L_K(E_F) \rightarrow L_K(E)$  was defined in [5] and in Proposition 1 of that paper it was shown that  $\text{im}(\theta)$  contains  $F \cup F^*$  and  $\{r(e) : e \in F\}$ .

The following observations will be useful in our proof: If  $F$  is a finite set of edges in  $E$  then a path  $p = e_1 e_2 \cdots e_n$  with  $e_i \in F$  is a cycle in  $E$  if and only if  $\bar{p} = (e_1, e_2) \cdots (e_n, e_1)$  is a cycle in  $E_F$ . Moreover, if  $C_1 \geq \cdots \geq C_k$  is a chain of cycles in  $E$  where the edges in all of the  $C_i$  belong to  $F$  then  $\bar{C}_i \geq \cdots \geq \bar{C}_k$  is a chain of cycles in  $E_F$ . Also suppose  $C_1 = e_1 e_2 \cdots e_n$  and  $C_2 = f_1 f_2 \cdots f_m$  are two cycles in  $E$  with  $F' = \{e_1, \dots, e_n, f_1, \dots, f_m\}$  then  $C_1, C_2$  will have a common vertex  $s(e_1) = s(f_1)$  in  $E$  if and only if, in the graph  $E_{F'}$ ,  $(e_1, e_2) \cdots (e_{n-1}, e_n)(e_n, f_1)(f_1, f_2) \cdots (f_{m-1}, f_m)(f_m, e_1)$  is a cycle sharing common vertices with the cycle  $(e_1, e_2) \cdots (e_n, e_1)$  and with the cycle  $(f_1, f_2) \cdots (f_m, f_1)$ .

We begin with the following easy Lemma.

**Lemma 5.1** *Let  $E$  be an arbitrary graph. If  $A$  is the (graded) ideal generated by the set  $X$  of all the acyclic vertices in  $E$  and  $N$  is the (graded) ideal generated by the set  $Y$  of vertices in all the cycles with no exits in  $E$ , then  $A \cap N = 0$ .*

**Proof.** Now both the ideals  $A$  and  $N$  possess local units and so  $A \cap N = AN$ . So it is enough to show that  $AN = 0$ . Suppose  $a = \sum k \alpha_i \beta_i^* \in A$  so that  $r(\alpha_i) = r(\beta_i) \in X$  and let  $b = \sum l_j \gamma_j \delta_j^* \in N$  so that  $r(\gamma_j) = r(\delta_j) \in Y$ . If  $ab \neq 0$ , then for some  $i, j$ ,  $\alpha_i \beta_i^* \gamma_j \delta_j^* \neq 0$  which implies  $\beta_i^* \gamma_j \neq 0$ . This means that either  $\beta_i = \gamma_j p$  or  $\gamma_j = \beta_i q$  where  $p, q$  are some paths. This leads to a contradiction for the following reasons. If  $\beta_i = \gamma_j p$ , then  $p$  gives rise to an exit for the no exit cycle containing  $r(\gamma_j)$ , a contradiction. If  $\gamma_j = \beta_i q$ , then  $q$  is a path from  $r(\beta_i)$  to the vertex  $r(\gamma_j)$  which sits on a cycle, contradicting the fact that  $r(\beta_i)$  is an acyclic vertex. Hence  $A \cap N = AN = 0$ . ■

We are now ready to prove the main theorem of this section.

**Theorem 5.2** *Let  $E$  be an arbitrary graph,  $K$  be any field and let  $L = L_K(E)$ . Then the following conditions are equivalent:*

- (i)  $L$  has finite GK-dimension  $\leq m$ , where  $m$  is a nonnegative integer;
- (ii) The relation  $\geq$  defines a partial order in the set  $P$  of all cycles in  $E$  and there is a non-negative integer  $m$  such that every chain in  $P$  has length at most  $m$ ;
- (iii)  $L$  is the union of a finite chain of graded ideals

$$0 \leq I_0 < I_1 < \cdots < I_m = L$$

where  $m$  is a fixed non-negative integer,  $I_0$  (may be zero) is von Neumann regular and, for each  $j$  with  $0 \leq j \leq m-1$ ,  $I_{j+1}/I_j$  is a direct sum of a von Neumann regular ring and/or direct sums of matrix rings of the form  $M_{\Lambda_j}(K[x, x^{-1}])$  where  $\Lambda_j$  are arbitrary index sets.

**Proof.** Assume (i). Suppose, by way of contradiction,  $E$  contains two distinct cycles  $C, C'$  having a common vertex. Let  $F$  be the set of all edges belonging to  $C$  and  $C'$ . Then the (finite) graph  $E_F$  will contain two cycles with a common vertex and so, by Theorem 5 of [7],  $L_K(E_F)$  has infinite GK-dimension. Then the subalgebra  $\text{im}(\theta) \cong L_K(E_F)$  (See Result (b)) and hence  $L$  will have infinite GK-dimension, a contradiction. So distinct cycles in  $E$  have no common vertex. This makes the relation  $\geq$  antisymmetric and hence a partial order in the set of all the cycles in  $E$ . If  $m = GK - \dim(L) = 0$ , then  $E$  must be acyclic. Because, if there is a cycle  $c$  in  $E$  based at a vertex  $v$ , then consider  $V := Kv \oplus Kc$ . Since  $\dim(V^n) \geq n$ , this forces  $GK - \dim(L) \geq 1$ , a contradiction. Thus  $E$ , being acyclic, trivially satisfies Condition (ii). So Assume  $m \geq 1$ . Suppose there is a chain of cycles  $C_1 \geq \dots \geq C_d$  of length  $d > m$  in  $E$ . For each  $i$ , let  $\gamma_i$  be a path connecting a fixed vertex on  $C_i$  to a fixed vertex on  $C_{i+1}$ . If  $F$  is the set of all the edges in the cycles  $C_1, \dots, C_d$  and in the paths  $\gamma_1, \dots, \gamma_{d-1}$ , then the subalgebra  $\text{im}(\theta) \cong L_K(E_F)$  will have, by Theorem 5 of [7], GK-dimension  $\geq 2d - 1$  which is  $> m$ . Clearly then the GK-dimension of  $L$  is  $> m$ , a contradiction. This proves (ii).

Assume (ii). We prove (iii) by induction on  $m$ . Suppose  $m = 0$ . This means that the graph  $E$  contains no cycles and consequently  $L$  is von Neumann regular, by Theorem 1 of [5]. So Condition (iii) holds with  $L = I_0$ . Suppose  $m \geq 1$  and assume that we have shown that  $L$  satisfies Condition (iii) if the upper bound for the lengths of chains of cycles in  $E$  is  $m - 1$ . Note that if a cycle  $C$  is a minimal element in the (artinian) partially ordered set  $P$  of cycles under  $\geq$  in  $E$ , then either  $C$  has no exits or for each exit  $e$  for  $C$ ,  $r(e)$  is an acyclic vertex. Let  $I_0$  be the graded ideal generated by the set  $\{r(e) : e \text{ an exit for a minimal cycle in } P\}$ . Since each such  $r(e)$  is an acyclic vertex,  $I_0$  is von Neumann regular, by Corollary 4.5. If  $H_0 = I_0 \cap E^0$ , then  $I_0 = I(H_0, \emptyset)$  where  $\emptyset$  is the empty set and in  $E \setminus (H_0, \emptyset)$  the minimal cycles in the poset of cycles have no exit. Now  $L/I_0 \cong L_K(E \setminus (H_0, \emptyset))$ . Identifying  $L/I_0$  with  $L_K(E \setminus (H_0, \emptyset))$ , let  $I_1/I_0$  denote the (graded) ideal generated by all the acyclic vertices and the vertices in all the cycles without exits in  $E \setminus (H_0, \emptyset)$ . By Proposition 3.7 of [3] and further Corollary 4.5 and Lemma 5.1 above,  $I_1/I_0$  is a direct sum of a von Neumann regular ring and a direct sum of matrix rings of the form  $M_{\Lambda^{(1)}}(K[x, x^{-1}])$  with  $\Lambda^{(1)}$  arbitrary index sets. Let  $H_1 = I_1 \cap E^0$ . Then  $L/I_1 \cong L_K(E \setminus (H_1, \emptyset))$ . Since  $(E \setminus (H_1, \emptyset))^0 = E^0 \setminus H_1 \cup \{u' : u \in B_{H_1}\}$  and since the  $u'$  are all sinks in  $E \setminus (H_1, \emptyset)$ , the maximum length of chains in  $E \setminus (H_1, \emptyset)$  is  $m - 1$ . So by induction,  $L/I_1$  is the union of a chain of graded ideals which we conveniently write as

$$0 < I_2/I_1 < \dots < I_m/I_1 = L/I_1$$

where, for  $1 \leq j \leq m - 1$ ,  $(I_{j+1}/I_1)/(I_j/I_1)$  is a direct sum of a von Neumann regular ring and direct sums of matrix rings of the form  $M_{\Lambda^{(j)}}(K[x, x^{-1}])$  with  $\Lambda^{(j)}$  arbitrary index sets. From this we immediately obtain the needed chain for  $L$  with the stated properties of Condition (iii).

Assume (iii). We wish to prove (ii). For each  $j$ , let  $H_j = I_j \cap E^0$  and so  $E^0$  is the union of the chain of hereditary saturated sets  $\emptyset \subseteq H_0 \subset H_1 \subset$

$\cdots \subset H_m = E^0$ . Suppose, by way of contradiction,  $E$  contains two cycles  $g, h$  having a common vertex  $v$ . Let  $j \geq 0$  be the smallest integer such that  $v \notin H_j$ . Then  $v \in H_{j+1}$  and in that case  $g^0, h^0 \subset H_{j+1} \setminus H_j$ . Now  $I_{j+1}/I_j$  is a direct sum a von Neumann regular ring and direct sums of matrix rings of the form  $M_{\Lambda^{(j)}}(K[x, x^{-1}])$  with  $\Lambda^{(j)}$  arbitrary index sets and so  $I_{j+1}/I_j$  has finite GK-dimension. Also identifying  $L/I_j$  with  $L_K(E \setminus (H_j, \emptyset))$ , we get an isomorphism  $I_{j+1}/I_j \cong L_K(\overline{E}(H_{j+1} \setminus H_j, \emptyset))$  by (Theorem 6.1, [26]). From the proof of (i)  $\implies$  (ii) it is then clear that distinct cycles in  $\overline{E}(H_{j+1} \setminus H_j, \emptyset)$  have no common vertex. But this contradicts the assumption that the cycles  $g, h$  have a common vertex  $v$  in  $\overline{E}(H_{j+1} \setminus H_j, \emptyset)$ . Hence no two distinct cycles in  $E$  will have a common vertex. Suppose there is a chain of cycles  $C_1 \geq C_2 \geq \cdots \geq C_d$  with  $d > m$  and the cycle  $C_d$  is based at a vertex  $u$ . As before if  $j \geq 0$  is the smallest integer such that  $u \notin H_j$ , then  $u \in H_{j+1} \setminus H_j$  and so  $(C_d)^0 \subset H_{j+1} \setminus H_j$ . If also  $(C_{d-1})^0 \subset H_{j+1} \setminus H_j$ , then the GK-dimension of  $I_{j+1}/I_j$  will be  $\geq 2$ . Because, if  $F$  is the finite set of edges on the cycles  $C_d, C_{d-1}$  and on a path connecting these two cycles, then the subalgebra of  $I_{j+1}/I_j$  isomorphic to  $L_K(FE)$  will have GK-dimension  $\geq 2$ , by Theorem 5 of [7], contradicting the fact that  $I_{j+1}/I_j$  has GK-dimension  $\leq 1$ . So  $(C_{d-1})^0 \not\subset H_{j+1} \setminus H_j$ . Proceeding like this we reach a conclusion that  $(C_{d-(m-j)})^0 \not\subset H_{j+(m-j)} = H_m = E^0$ , a contradiction. Hence every chain of cycles in  $E$  must have length at most  $m$  and this proves (ii).

Assume (ii). It was shown in (Proposition 2, [5])  $L = \varinjlim B(S)$  where  $S$  varies over all the finite subsets of  $L$  and  $B(S)$  is a subalgebra generated by  $S$ . Moreover, each  $B(S)$  is isomorphic to  $L_K(E_F) \oplus V$  where  $F$  is a finite set of edges on the paths that show up in the representation of the elements of  $S$  as  $K$ -linear combinations of monomials and  $V$  is a finite dimensional  $K$ -algebra (see Proposition 2, [5]). By hypothesis and the remarks in Result (b) above, the graph  $E_F$ , for every finite set of edges  $F$ , satisfies Condition (ii). Hence, by Theorem 5 of [7], each  $L_K(E_F)$  has GK-dimension  $\leq m$ . This means each  $B(S)$  and hence  $L$  has GK-dimension  $\leq m$ . This proves (i). ■

REMARK: We wish to point out that, for a finite graph  $E$ , the chain of ideals in Condition (3) of Theorem 3.10 becomes finite with the successive quotients direct sums of matrix rings over  $K$  and/or  $K[x, x^{-1}]$  and this (for the finite graph  $E$ ) has been shown in [8] to be a necessary condition for  $L_K(E)$  to have finite GK-dimension and that in [24], it is also shown to be a sufficient condition for finite GK-dimension of  $L_K(E)$ .

**Acknowledgement 5.3** *I am deeply grateful to Gene Abrams for giving me the benefit his preprint [4] containing the crucial Lemmas that were used in the proof of Corollary 3.5 and for useful discussions.*

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